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Bases in Function Spaces on Compact Sets

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Abstract. This is a brief history, covering the twentieth century, of spline bases on cubes, and an exposition of constructing bases in classical function spaces over compact smooth finite dimensional manifolds.

§1. Introduction

The aim of this paper is to present an overview on some twentieth century developments in the theory of spline bases. We start by recalling some of the relevant notions on bases in Banach spaces (for more details see e.g. [1,30]). For simplicity we are going to stay within the real Banach spaces. An abstract Banach space X with the norm $\|\cdot\|_X$ is denoted as $[X, \|\cdot\|_X]$. The sequence $(x_n, n=0,1,\ldots)$ in $[X, \|\cdot\|_X]$ is called a basis in X if to each $x \in X$ there is a unique sequence of scalars $\underline{a} = (a_n, n=0,1,\ldots)$ such that

$$x = \sum_{n=0}^{\infty} a_n\, x_n. \tag{1}$$

There are unique linear functionals $(x_n^*) \subset X^*$ such that $a_n = x_n^*(x)$. The system $(x_0, x_1, \ldots; x_0^*, x_1^*, \ldots)$ is biorthogonal i.e. $x_k^*(x_i) = \delta_{k,i}$. The basis (x_n) is unconditional if for each $x \in X$ the series in the right hand side of (1) converges unconditionally. Now, denote by $\mathcal{A}$ the set of all $\underline{a}$ appearing in (1) while x is running through X. The linear space $\mathcal{A}$ becomes a Banach space linearly isomorphic to X with the norm

$$\|\underline{a}\|_{\mathcal{A}} = \sup_{n\geq 0} \|\sum_{i=0}^{n} a_i\, x_i\|_X. \tag{2}$$

The Banach space $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is customarily called the coefficient space. Introducing the basis constant, which by the Banach-Steinhaus theorem is finite,

$$\beta = \sup_{\|x\|_X \leq 1}\, \sup_{n\geq 0} \|\sum_{i=0}^{n} a_i\, x_i\|_X, \tag{3}$$

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we obtain the equivalence of norms

$$\|x\|_X \leq \|a\|_{\mathcal{A}} \leq \beta \|x\|_X. \tag{4}$$

Thus, *every Banach space with a basis is linearly isomorphic to a sequence space.* However, depending on the space X and on the particular basis, the corresponding sequence space may be of little use. Sometimes it helps to consider equivalent basis in X. Two basis $(x_n) \subset X$ and $(x'_n) \subset X'$ are said to be equivalent if $\mathcal{A} = \mathcal{A}'$. In case of equivalent bases we conclude that $\|\underline{a}\|_{\mathcal{A}} \sim \|\underline{a}\|_{\mathcal{A}'}$ for $\underline{a} \in \mathcal{A}$. Now, we may describe the program of the paper. For a given compact C^∞ finite dimensional manifold M and for given order of smoothness m, we are going to describe the construction of biorthogonal system of functions of class C^m over M such that the system itself is a basis in $VMO(M)$, $H_1(M)$ and and in the whole scale of Sobolev $W_p^k(M)$, $-m \leq k \leq m$, and of Besov spaces $B_{p,q}^s(M)$, $-m < s < m$, with $1 \leq p, q \leq \infty$. At the same time the dual system is going to be a basis in the same scale of function spaces with the corresponding spaces VMO, H_1, W and B replaced by $\overset{\circ}{VMO}, \overset{\circ}{H}_1, \overset{\circ}{W}$ and $\overset{\circ}{B}$, respectively. The constructed system of functions (or its dual) is always an unconditional basis whenever the space admits an unconditional basis. Moreover, for the constructed basis, we are able to describe the coefficient spaces in case of the BMO and Besov spaces. The duality questions will be treated at the same time. The main idea of the general construction was announced by T. Figiel and the author at the Gdańsk 1979 conference: *Approximation and function spaces* (cf. [14]), and then carried out in the subsequent papers [10,11,15,16,17].

The material is arranged as follows: Sections 2 presents historical remarks on the Haar, Faber-Schauder, Franklin and spline systems; Section 3 treats function spaces and bases with boundary conditions on the cube; Section 4 describes the reduction of function spaces and bases from manifolds to the cubes with boundary conditions.

It is encouraging, that in recent years, the ideas of the constructions from [16,17] stimulated works on modifications of the decomposition of the function spaces on smooth compact manifolds into standard spaces, and also on constructing new bases in the standard spaces. The new investigations of W. Dahmen and R. Schneider as they were presented at this Saint Malo conference (see also [19]) are very promising as they show that these constructions can be applied to treat singular operators on manifolds both theoretically and numerically.

§2. The History of Haar, Faber-Schauder, Franklin and Spline Systems

At the very origin there is the construction of A. Haar (1909) [25] of a simple ONC (orthonormal and complete) system $\underline{\chi} = (\chi_n, n = 1, \ldots)$ on $I = [0, 1]$. The system $\underline{\chi}$ has the nice property that each continuous function has its Fourier-Haar series uniformly convergent on I. Here and later on, unless

otherwise stated, the orthogonality is understood with respect to the Lebesgue measure. The orthonormal Haar functions over I can be defined by means of a single function h, where

$$h(t) = \begin{cases} 1 & \text{for } -\frac{1}{2} < t \le 0, \\ -1 & \text{for } 0 < t \le \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Define for $j \ge 0$, $1 \le k \le 2^j$ and $n = 2^j + k$

$$\chi_1 = 1, \quad \text{and} \quad \chi_n(t) = \chi_{j,k}(t) = 2^{j/2} h(2^j(t - \frac{2k-1}{2^{j+1}})). \tag{5}$$

The Haar functions are piecewise constant and left continuous i.e. they are splines of order $r = 1$ (of degree $r - 1 = 0$). For later convenience for the support of χ_n we introduce the symbol $(n) = [\frac{k-1}{2^j}, \frac{k}{2^j}]$ (and let $t_n = \frac{2k-1}{2^{j+1}}$ denote the middle point). It was pointed out by J. Schauder (1927) [29] that the Haar system $\underline{\chi}$ is a basis in the Lebesgue space $L_p(I)$, $1 \le p < \infty$, with the basis constant equal to 1. Much more involved was the proof of R.E.A.C. Paley (1932) [27] (see also J. Marcinkiewicz (1937) [26]) that the Haar system is an unconditional basis in each $L_p(I)$, $1 < p < \infty$. For a real variable proof of this property, we refer e.g. to Ch. Watari (1964) [31]. The unconditional basic constant for the Haar system in $L_p(I)$ appears to be equal to $\max(p, p') - 1$ where $1/p + 1/p' = 1$ (see e.e. D.L. Burkholder [5]). The extensively investigated martingale theory covers many results on the Haar system, but it is not very related to our subject, and will not be discussed here (see e.g. [23]).

To construct the Faber-Schauder, Franklin and more general spline systems, it is convenient to introduce the following operations on sequences of functions. For a given sequence $\underline{\psi} = (\psi_n, n = 1, 2, \ldots)$ of integrable functions on I, we define

$$\mathbf{G}\underline{\psi} = (1, \mathrm{G}\psi_n, n = 1, 2, \ldots) \quad \text{and} \quad \mathbf{G}_0\underline{\psi} = (\mathrm{G}\psi_n, n = 1, 2, \ldots),$$

where $\mathrm{G}f(t) = \int_0^t f(s)\,ds$. If in addition, the functions in $\underline{\psi}$ are linearly independent, then the result of the Gram orthogonalization process applied to $\underline{\psi}$ is denoted by $\mathbf{O}\underline{\psi}$. It is assumed in this definition that so obtained orthogonal set is normalized in $L^2(I)$.

The Faber-Schauder system can now be obtained from the Haar system by the operation $\mathbf{G}$

$$\underline{\phi} = (\phi_n, n \ge 0) = \mathbf{G}\underline{\chi}. \tag{6}$$

The Faber-Schauder functions are continuous splines of order 2. It was proved by G. Faber (1910) [21] (see also J. Schauder (1927) [29]) that this system is a basis in the space of continuous functions $[C(I), \|\cdot\|_\infty]$. In this case the

basis constant is again equal to 1 and the basis itself is interpolating at dyadic points of I.

The orthonormal set constructed by Ph. Franklin (1928) [22] can now be defined as the result of application of the operation $\mathbf{O}$ to the Schauder system

$$\underline{f} = (f_n, n = 0, 1, \ldots) = \mathbf{O}\underline{\phi}. \tag{7}$$

These functions are again continuous splines of order $r = 2$. Ph. Franklin proved in [22] that $\underline{f}$ is a basis in $[C(I), \|\cdot\|_\infty]$. For an elegant proof that the Franklin system is a basis in $C(I)$ and in $L_p(I)$, $1 \leq p < \infty$, we refer to [6]. Using the same idea as in [31], S. V. Botchkarev (1974, 1975) [3, 4] proved the unconditionality of the Franklin system in each $L_p(I)$, $1 < p < \infty$. There is an extensive literature on the pointwise behavior of the Franklin series, but we mention only the expository article by G. G. Gevorkyan [24].

The operation $\mathbf{G}$ increases the order r of splines and the order of their smoothness by 1, and $\mathbf{O}$ preserves these orders. We may repeat this two step process starting now with the orthonormal Franklin system and then repeat it again and again. In general, for $r \geq 1$ we use the notation

$$\underline{f}^{(r)} = (f_n^{(r)}, n > 1 - r) \quad \text{and} \quad \underline{\phi}^{(r,1)} = (\phi_n^{(r,1)}, n > -r) = \mathbf{G}\underline{f}^{(r)}$$

and

$$\underline{f}^{(r+1)} = \mathbf{O}\underline{\phi}^{(r,1)}.$$

Consequently, we have for the order $r \geq 1$ the following inductive formula for the spline ONC system on I

$$\underline{f}^{(r+1)} = \mathbf{O} \circ \mathbf{G}\underline{f}^{(r)} \quad \text{with} \quad \underline{f}^{(1)} = \underline{\chi}. \tag{8}$$

In particular in this notation $\underline{\phi}^{(1,1)} = \underline{\phi}$ and $\underline{f} = \underline{f}^{(2)}$. It was proved in [7] that $\underline{\phi}^{(2,1)}$ is a basis in $[C(I), \|\cdot\|_\infty]$ and by J. Radecki (1970) [28] that $\underline{f}^{(3)}$ is a basis in $[C(I), \|\cdot\|_\infty]$ and in each $L_p(I)$, $1 \leq p < \infty$. The proof that $\underline{f}^{(r)}$ for arbitrary $r \geq 1$ is a basis in $C(I)$ and in $L_p(I)$ follows from the work of J. Domsta (1972) [20] (see also [12]).

From the construction of the ONC system $\underline{f}^{(r)}$ it follows that its first r elements $f_{2-r}^{(r)}, \ldots, f_1^{(r)}$ are simply the orthonormal Legendre polynomials on I; the degree of $f_i^{(r)}$ is $i + r - 2$. Now, with each $r \geq 1$ we associate a family of spline systems

$$\underline{f}^{(r,k)} = (f_n^{(r,k)}, n > |k| + 1 - r) \quad \text{with} \quad -r \leq k < r, \tag{9}$$

where

$$f_n^{(r,k)} = \begin{cases} \mathrm{D}^k f_n^{(r)} & \text{for } 0 \leq k < r, \\ \mathrm{H}^{-k} f_n^{(r)} & \text{for } -r \leq k < 0, \end{cases} \tag{10}$$

and $\mathrm{D}f(t) = \frac{d}{dt}f(t)$ and $\mathrm{H}f(t) = \int_t^1 f(s)\,ds$. Since D is inverse to G and H is adjoint to G in $L^2(I)$, it follows that for $|k| < r$,

$$(f_n^{(r,k)}, f_m^{(r,-k)}) = \delta_{n,m} \quad \text{for} \quad n, m > |k| + 1 - r. \tag{11}$$

Equally important are spline ON systems defined by formula similar to (8) with $\mathbf{G}$ replaced by $\mathbf{G}_0$, i.e.

$$\underline{g}^{(r+1)} = \mathbf{O} \circ \mathrm{G}_0 \underline{g}^{(r)} \quad \text{with} \quad \underline{g}^{(1)} = \underline{\chi}. \tag{12}$$

Here again, with each $r \geq 1$ we associate a family of spline systems

$$\underline{g}^{(r,k)} = (g_n^{(r,k)}, n \geq 1) \quad \text{with} \quad -r \leq k < r, \tag{13}$$

where

$$g_n^{(r,k)} = \begin{cases} \mathrm{D}^k g_n^{(r)} & \text{for } 0 \leq k < r, \\ \mathrm{H}^{-k} g_n^{(r)} & \text{for } -r \leq k < 0; \end{cases} \tag{14}$$

and as before we have for $|k| < r$,

$$(g_n^{(r,k)}, g_m^{(r,-k)}) = \delta_{n,m} \quad \text{for} \quad n, m \geq 1. \tag{15}$$

In what follows in this section we denote by $\underline{F}^{(r)}$ either $\underline{f}^{(r)}$ or $\underline{g}^{(r)}$. Since the family $\{\underline{F}^{(r,k)}, -r \leq k < r\}$ of spline systems is the main ingredient in the construction presented in the next section, it is natural to recall now its basic properties. Notice that the elements of $\underline{F}^{(r,k)}$ are indexed by $n \geq n(k, F)$, where $n(k, f) = |k| + 2 - r$ and $n(k, g) = 1$.

For given r and k such that $|k| < r$ and for given $n \geq n(k, F)$, we have the kernel corresponding to the partial sum operator with index n

$$K_n^{(r,k)}(s,t) = \sum_{\nu = n(k,F)}^{n} F_\nu^{(r,-k)}(s) \cdot F_\nu^{(r,k)}(t) \quad \text{for} \quad s, t \in I. \tag{16}$$

The following exponential estimates (cf. [8,13,17]) play a fundamental role in our construction. There are two constants: $C = C_r < \infty$ and $q = q_r$, $0 < q < 1$, such that for $|k| < r$ we have

$$|K_n^{(r,k)}(s,t)| \leq C \cdot (n + r) \cdot q^{(n+r)|s-t|} \quad \text{for} \quad s, t \in I, \tag{17}$$

and

$$|F_n^{(r,k)}(t)| \leq C \cdot (n + r)^{k + \frac{1}{2}} \cdot q^{(n+r)|t - t_n|} \quad \text{for} \quad t \in I, \tag{18}$$

where t_n has been defined earlier as the middle point of (n). Now, the biorthogonality (11), (15) and (18) imply for $1 \leq p \leq \infty$, $|k| < r$, and for any real sequence $(a_n, 2^j < n \leq 2^{j+1})$, $(j \geq 0)$, the equivalence

$$\| \sum_{2^j < n \leq 2^{j+1}} a_n \cdot F_n^{(r,k)} \|_p \sim 2^{j(k + \frac{1}{2} - \frac{1}{p})} \cdot (\sum_{2^j < n \leq 2^{j+1}} |a_n|^p)^{\frac{1}{p}}. \tag{19}$$

Moreover, it follows that

$$\| \sum_{2^j < n \leq 2^{j+1}} a_n \cdot F_n^{(r,k)} \|_p \sim \| \sum_{2^j < n \leq 2^{j+1}} |a_n \cdot F_n^{(r,k)}| \|_p, \tag{20}$$

where the positive constants in the equivalences $\sim$ in (19) and (20) depend on r only.

Now, as one of the consequences of (17) and (18), we obtain

Theorem 1. *For given r, k, $|k| < r$ and p, $1 \le p < \infty$, the system $\underline{F}^{(r,k)}$ is a basis in $L_p(I)$. Moreover, for $1 < p < \infty$, each of the systems is an unconditional basis in $L_p(I)$, and all of them are equivalent bases in this space. Moreover, $\underline{F}^{(r,k)}$ is a basis in $C(I)$ for each k, $0 \le k \le r-2$.*

§3. The Standard Spaces over Cubes

We start with general setup which will be needed in the following sections. Let the dimension d be fixed, and let M be a compact C^∞ d-dimensional manifold (d-manifold). For simplicity, we assume here that M has no boundary. We denote by μ one of the measures which locally is of the form $d\mu = h\,dx$ where h is positive C^∞ function. A closed set $Q \subset M$ is said to be a d-cube if it is diffeomorphic to the standard cube $[0,1]^d$. A compact set $K \subset M$ or $K \subset R^d$ is said to be proper if it can be viewed locally as an epigraph of a lipschitzian function of $d-1$ variables (cf. Def. 3.1 in [16]). We are going to discuss function spaces $\mathcal{F}(K)$ over a proper subsets K, in particular the Sobolev spaces with $\mathcal{F} = W_p^m$ and the Besov spaces with $\mathcal{F} = B_{p,q}^\alpha$. In the Sobolev space $W_p^m(K)$, $K \subset R^d$, $1 \le p \le \infty$, $m \ge 0$, we shall use the norm

$$\|f\|_p^{(m)}(K) = \sum_{|\alpha| \le m} \|D^\alpha f\|_p(K). \tag{21}$$

Clearly, $W_p^0(K) = L_p(K)$ and we denote by $W_\infty^m(K)$ the space $C^m(K)$. Moreover, the space of equivalence classes of measurable functions over K equipped with the topology of convergence in measure is denoted by $L_0(K)$. In order to define $W_p^k(K)$ for $k < 0$, we introduce $\overset{\circ}{W}{}_p^m(K)$ for each $m \ge 0$ as the closure in the norm (21) of smooth functions f such that $\overline{\operatorname{supp} f} \subset \operatorname{int} K$. For $1 \le p \le \infty$, $k < 0$ and for $g \in W_{p'}^0(K)$ put

$$\|g\|_{p'}^{(k)}(K) = \sup\Big\{\Big|\int_K fg\,dx\Big| : \|f\|_p^{(-k)}(K) \le 1,\ f \in \overset{\circ}{W}{}_p^{-k}(K)\Big\}, \tag{22}$$

where $p' = p/(p-1)$ for $1 < p < \infty$ and $1' = \infty$, $\infty' = 1$. Now, the completion of $W_{p'}^0(K)$ in the norm (22) defines the space $W_{p'}^k(K)$.

Let now $I = [0,1]$, $Q = I^d$ and let $\boldsymbol{Z}$ be a boundary set i.e. a set which is a union of $(d-1)$-dimensional faces of Q. To each pair $\{\mathcal{F}(Q), \boldsymbol{Z}\}$ we associate a subspace of $\mathcal{F}(Q)$ of functions which are vanishing on $\boldsymbol{Z} \subset \partial Q$ in the sense described below. To each $\boldsymbol{Z}$ there are unique $Z_i \subset \partial I$, $i = 1, \dots, d$, such that

$$\boldsymbol{Z} = Q \setminus (I \setminus Z_1) \times \ldots \times (I \setminus Z_d). \tag{23}$$

Now, define for each $\boldsymbol{Z}$ the parallelepiped

$$Q_{\boldsymbol{Z}} = I_{Z_1} \times \ldots \times I_{Z_d}, \tag{24}$$

where

$$I_Z = \begin{cases} [0,1] & \text{for } Z = \emptyset, \\ [-1,1] & \text{for } Z = \{0\}, \\ [0,2] & \text{for } Z = \{1\}, \\ [-1,2] & \text{for } Z = \{0,1\}. \end{cases} \tag{25}$$

If $f \in L_0(Q)$, we denote by f_Z the element of $L_0(Q_Z)$ such that $f_Z|_Q = f$ and $f_Z = 0$ on $Q_Z \setminus Q$.

Definition 2. *For given integer k and $1 \le p \le \infty$, put*

$$\|f\|_p^{(k)}(Q)_Z = \|f_Z\|_p^{(k)}(Q_Z).$$

Now, if $k \geq 0$, define

$$W_p^k(Q)_Z = \{f \in W_p^0(Q) : \ f_Z \in W_p^k(Q_Z)\},$$

and if $k < 0$ then introducing $W_0 = \{f \in W_p^0(Q) : \ f_Z \in W_p^k(Q_Z)\}$, define

$$W_p^k(Q)_Z = \textit{completion of } W_0 \textit{ in the norm } \|f\|_p^{(k)}(Q)_Z.$$

The spaces $[W_p^k(Q)_Z, \|\cdot\|_p^{(k)}(Q)_Z]$ are called standard.

Notice that for $k \geq 0$ the set $\{f_Z : \ f \in W_p^k(Q)_Z\}$ is a closed subspace of $W_p^k(Q_Z)$, and by Definition 2 the map $f \mapsto f_Z$ is an isometry. Now let $k < 0$. In this case the map $f \mapsto f_Z$ extends to an isometry of $W_p^k(Q)_Z$ into $W_p^k(Q_Z)$. Thus $W_p^k(Q)_Z$ is always complete, and the image of the map $f \mapsto f_Z$ is a closed subspace of $W_p^k(Q_Z)$. We have constructed in [16], using the formulae (23) and (24) and the generalized Hestenes extension operators, a bounded projection onto this subspace.

Proposition 3. *Let $m \geq 1$ and the boundary set $Z \subset \partial Q$ be given. Then there are a continuous linear operator P in $L_0(Q_Z)$ and $C < \infty$ such that P projects $L_0(Q_Z)$ onto $\{f : \ f = 0$ a.e. on $Q_Z \setminus Q\}$ and for $1 \le p \le \infty$ we have*

$$\|Pf\|_p^{(k)}(Q_Z) \le C\,\|f\|_p^{(k)}(Q_Z) \ \textit{for } f \in W_p^k(Q_Z),\ |k| \le m. \tag{26}$$

Thus, P projects $W_p^k(Q_Z)$ onto a subspace which is via the map $f \mapsto f_Z$ linearly isomorphic to $W_p^k(Q)_Z$.

Now, for $k \le 0$ and $1 \le p \le \infty$ we define the bilinear form

$$g^*(f) = \int_Q fg\,dx \quad \text{for} \quad g^* \in (W_{p'}^{-k}(Q)_{Z'})^*,\ g \in W_p^0(Q). \tag{27}$$

Proposition 4. *Let $k \le 0$ and $1 \le p \le \infty$ be given. Then the map*

$$g \mapsto g^* : W_p^0(Q) \to (W_{p'}^{-k}(Q)_{Z'})^*$$

defined in (27) extends to a linear isomorphism of $W_p^k(Q)_Z$ onto a subspace of $(W_{p'}^{-k}(Q)_{Z'})^*$.

Now suppose real s and $1 \le p, q \le \infty$ are given. Moreover, let K be a proper set. Then for any integers k, l such that $l < s < k$, we have the real interpolation formula for the Besov space (with $\theta = (s-l)/(k-l)$)

$$B_{p,q}^s(K) = \left(W_p^l(K), W_p^k(K)\right)_{\theta,q}. \tag{28}$$

For $f \in B_{p,q}^s(K)$ the norm is denoted by $\|f\|_{p,q}^{(s)}(K)$. The Besov space over Q with $l = 0 < s < k$ and corresponding to the boundary set $Z \subset \partial Q$ is now defined by the formula

$$B_{p,q}^s(Q)_Z = \{f \in W_p^0(Q) : \ f_Z \in B_{p,q}^s(Q_Z)\}. \tag{29}$$

Moreover, let us define for $f \in B_{p,q}^s(Q)_Z$

$$\|f\|_{p,q}^{(s)}(Q)_Z = \|f_Z\|_{p,q}^{(s)}(Q_Z). \tag{30}$$

The Besov space $[B_{p,q}^s(Q)_Z, \|\cdot\|_{p,q}^{(s)}(Q)_Z]$ will be called standard as well. Notice, that $B_{p,q}^s(Q)_Z \subseteq B_{p,q}^s(Q)$, but it may be not a closed subset of $B_{p,q}^s(Q)$.

Proposition 5. *Let the parameters l, k, θ, s, p, q be given as for (28), and let*

$$\mathcal{F}(Q)_Z = (\mathcal{F}_0(Q)_Z, \mathcal{F}_1(Q)_Z)_{\theta,q} \text{ where } \mathcal{F}_0 = W_p^l,\ \mathcal{F}_1 = W_p^k. \tag{31}$$

Then, $\mathcal{F}(Q)_Z = B_{p,q}^s(Q)_Z$ for $s > 0$ and for $s < 0$ the space $\mathcal{F}(Q)_Z$ is naturally identified with the closure of W_p^0 in $(B_{p',q'}^{-s}(Q)_{Z'})^*$.

The proof is based on the existence of the projection P in Proposition 3, and on the general properties of the real interpolation spaces (see [2, 16]).

Corollary 6. *Suppose we are given real numbers s, $1 \le p, q \le \infty$, an integer k, and a boundary set $Z \subseteq \partial Q$ of the cube Q. Then the standard spaces $\mathcal{F}(Q)_Z$ are well defined for $\mathcal{F} = W_p^k$ or $B_{p,q}^s$. Moreover, if $l < s < k$, then formula (31) takes place.*

In the last part of Section 3 we are going to present a construction of spline bases in the $\mathcal{F}(Q)_Z$ spaces. Actually, according to (31), it is sufficient to do it for the Sobolev spaces $W_p^k(Q)_Z$.

We start with the case of dimension $d = 1$. To each $Z \subset \partial I$ and for an integer $m = r - 2 \ge 0$ a spline system is defined as follows:

$$F_n^{(m)}(t; Z) = \begin{cases} f_n^{(2r,r)}(t) & \text{if } Z = \emptyset \text{ and } n \ge 2-r, \\ f_n^{(2r,-r)}(t) & \text{if } Z = \{0,1\} \text{ and } n \ge 2-r, \\ g_n^{(2r,r)}(t) & \text{if } Z = \{0\} \text{ and } n \ge 1, \\ g_n^{(2r,-r)}(t) & \text{if } Z = \{1\} \text{ and } n \ge 1; \end{cases} \tag{32}$$

where the $\underline{f}^{(r,l)}$ and $\underline{g}^{(r,l)}$ are given as in (10) and (14), respectively. Moreover, let

$$n(Z) = n(Z,0) \quad \text{and} \quad n(Z,k) = \begin{cases} |k| + 2 - r & \text{if } Z = \emptyset, \\ |k| + 2 - r & \text{if } Z = \{0,1\}, \\ 1 & \text{if } Z = \{0\}, \\ 1 & \text{if } Z = \{1\}. \end{cases} \tag{33}$$

For simplicity let us write $\underline{F}^{(m)}(Z) = (F_n^{(m)}(\cdot\,;Z), n \geq n(Z))$. Notice that $n(Z,k) = n(Z',-k)$, and that the two systems $\underline{F}^{(m)}(Z)$ and $\underline{F}^{(m)}(Z')$, where $Z' = \partial I \setminus Z$, are dual, i.e. they are biorthogonal in the $L^2(I)$ scalar product

$$(F_i^{(m)}(\cdot\,;Z), F_j^{(m)}(\cdot\,;Z')) = \delta_{i,j} \quad \text{for} \quad i,j \geq n(Z). \tag{34}$$

Now, we introduce related family of biorthogonal systems indexed by k with $|k| \leq m$. Namely, for $j \geq n(Z,k)$, let

$$F_j^{(m,k)}(\cdot\,;Z) = \begin{cases} \mathrm{D}^k F_j^{(m)}(\cdot\,;Z) & \text{for } 0 \leq k \leq m, \\ \mathrm{H}^{-k} F_j^{(m)}(\cdot\,;Z) & \text{for } -m \leq k < 0; \end{cases} \tag{35}$$

and the biorthogonality for $|k| \leq m$ is as follows

$$(F_i^{(m,k)}(\cdot\,;Z), F_j^{(m,-k)}(\cdot\,;Z')) = \delta_{i,j} \quad \text{for} \quad i,j \geq n(Z,k). \tag{36}$$

Theorem 7. *For each $Z \subset \partial I$, $1 \leq p \leq \infty$, the system $\underline{F}^{(m)}(Z)$ is in $W_p^m(I)_Z$ and it is a basis (an unconditional basis if $1 < p < \infty$) in each $W_p^k(I)_Z$ for $k = 0, \ldots, m$. This means that it is a simultaneous basis (simultaneous unconditional basis if $1 < p < \infty$) in $[W_p^m(I)_Z, \|\cdot\|_p^{(m)}]$.*

Proof: To see how the proof works, let

$$P_n f(x;Z) = \sum_{n(Z) \leq j \leq n} (f, F_j^{(m)}(\cdot\,;Z')) F_j^{(m)}(x;Z) \quad \text{for} \quad f \in L_p(I). \tag{37}$$

Then we find that for $0 \leq k \leq m$,

$$D^k P_n f(x;Z) = P_n^{(k)}(D^k f)(x;Z) \quad \text{for} \quad f \in W_p^{(k)}(I)_Z, \tag{38}$$

where for $g \in W_p^0(I)$

$$P_n^{(k)}(g)(x;Z) = \sum_{j=n(Z,k)}^{n} (g, F_j^{(m,-k)}(\cdot\,;Z')) F_j^{(m,k)}(x;Z). \tag{39}$$

Now, Theorem 7 follows immediately from Theorem 1 by (38) and (39). □

Now we consider the case of dimension $d > 1$, with $Q = I^d$. Suppose we are given $\boldsymbol{Z} \subset \partial Q$. Then according to (23) the $Z_i \subset \partial I$, for $i = 1, \dots, d$, are determined. We are ready now to construct the tensor product basis corresponding to the boundary set $\boldsymbol{Z}$. Each function of the basis under construction is determined by an integer vector $\boldsymbol{j} = (j_1, \dots, j_d)$ satisfying the inequality $\boldsymbol{j} \geq \boldsymbol{n}(\boldsymbol{Z})$ with $\boldsymbol{n}(\boldsymbol{Z}) = (n(Z_1), \dots, n(Z_d))$ i.e. $j_i \geq n(Z_i)$ for $i = 1, \dots, d$.

Given the order of smoothness $m \geq 0$, we now define the $\boldsymbol{j}$'s function as follows:

$$F_{\boldsymbol{j}}^{(m)}(\boldsymbol{x}; \boldsymbol{Z}) = F_{j_1}^{(m)}(x_1; Z_1) \times \cdots \times F_{j_d}^{(m)}(x_d; Z_d), \tag{40}$$

where $\boldsymbol{x} = (x_1, \dots, x_d)$. The indices $\boldsymbol{j}$ are ordered in the rectangular way (cf. [12], p. 221).

Theorem 8. *The system $(F_{\boldsymbol{j}}^{(m)}(\cdot\,; \boldsymbol{Z}), \boldsymbol{j} \geq \boldsymbol{n}(\boldsymbol{Z}))$ in the rectangular ordering is a basis in $W_p^m(Q)_{\boldsymbol{Z}}$ for $1 \leq p \leq \infty$, and in addition it is unconditional in these spaces if $1 < p < \infty$.*

Our next goal is to modify the basis (40) in such a way that the elements of the new basis will be concentrated around the corresponding dyadic points in Q. To this end let us introduce in dimension one the following finite dimensional spline spaces:

$$S_\mu^m(Z) = \operatorname{span}\{F_j^{(m)}(\cdot\,; Z) : \ n(Z) \leq j \leq 2^\mu\} \quad \text{where} \quad \mu \geq 1. \tag{41}$$

Now, without going into details, we accept the Definition 10.17 of [17] of the new spline basis in $S_\mu^m(Z)$, i.e. of $F_{\mu,j}^{(m)}(\cdot\,; Z)$ with $n(Z) \leq j \leq 2^\mu$. The new basic functions for the standard space $W_p^m(Q)_{\boldsymbol{Z}}$ are now defined as follows. For convenience, let $D = \{1, \dots, d\}$, and let

$$N_0(\boldsymbol{Z}) = \{\boldsymbol{j} : \ n(Z_i) \leq j_i \leq 1 \text{ for } i \in D\},$$

and for every $e \subset D$, $e \neq \emptyset$, $\mu \geq 1$, let

$$N_{e,\mu}(\boldsymbol{Z}) = \{\boldsymbol{j} : \ 2^{\mu-1} < j_i \leq 2^\mu \text{ for } i \in e, \ n(Z_i) \leq j_i \leq 2^{\mu-1} \text{ for } i \in D \setminus e\}.$$

We also introduce

$$N_\mu(\boldsymbol{Z}) = \bigcup_{\emptyset \neq e \subset D} N_{e,\mu}(\boldsymbol{Z}).$$

Definition 9. *Let*

$$G_{\boldsymbol{j}}^{(m)}(\cdot\,; \boldsymbol{Z}) = F_{\boldsymbol{j}}^{(m)}(\cdot\,; \boldsymbol{Z}) \quad \text{for} \quad \boldsymbol{j} \in N_0(\boldsymbol{Z}),$$

and

$$G_{\boldsymbol{j}}^{(m)}(\cdot\,; \boldsymbol{Z}) = \bigotimes_{i \in e} F_{j_i}^{(m)}(\cdot\,; Z_i) \otimes \bigotimes_{i \in D \setminus e} F_{\mu,j_i}^{(m)}(\cdot\,; Z_i) \quad \text{for} \quad \boldsymbol{j} \in N_{e,\mu}(\boldsymbol{Z})$$

for any $e \subset D$, $e \neq \emptyset$, and $\mu \geq 1$.

Any ordering $\prec$ of the set of indices $\{\boldsymbol{j} : \boldsymbol{j} \geq \boldsymbol{n}(\boldsymbol{Z})\}$ is said to be regular if $\boldsymbol{j} \prec \boldsymbol{j}'$ for any $\boldsymbol{j} \in N_\mu$ and $\boldsymbol{j}' \in N_{\mu'}$ whenever $\mu < \mu'$ (cf. [10]). We also have the biorthogonality relation

$$(G_{\boldsymbol{j}}^{(m)}(\cdot\,;\boldsymbol{Z}), G_{\boldsymbol{j}'}^{(m)}(\cdot\,;\boldsymbol{Z}')) = \delta_{\boldsymbol{j}',\boldsymbol{j}}. \tag{42}$$

We can now state the result on 'universal basis' in standard spaces (cf. [10, 16])

Theorem 10. *Let $m \geq 0$ be a given order of smoothness. The system $(G_{\boldsymbol{j}}^{(m)}(\cdot\,;\boldsymbol{Z}), \boldsymbol{j} \geq \boldsymbol{n}(\boldsymbol{Z}))$ in the regular ordering is a basis in all the spaces $W_p^k(Q)_{\boldsymbol{Z}}$ for $0 \leq k \leq m$, $1 \leq p \leq \infty$, and in addition it is unconditional if $1 < p < \infty$. Moreover, for $\mu \geq 1$, $1 \leq p \leq \infty$, we have*

$$\|\sum_{\boldsymbol{j}\in N_\mu} a_{\boldsymbol{j}}\, G_{\boldsymbol{j}}^{(m)}(\cdot\,;\boldsymbol{Z})\|_p \sim 2^{\mu(1/2-1/p)d}\Big(\sum_{\boldsymbol{j}\in N_\mu} |a_{\boldsymbol{j}}|^p\Big)^{1/p}, \tag{43}$$

where the constants in the relation $\sim$ depend only on d and m.

Corollary 11. *The system $(G_{\boldsymbol{j}}^{(m)}(\cdot\,;\boldsymbol{Z}), \boldsymbol{j} \geq \boldsymbol{n}(\boldsymbol{Z}))$ in the regular ordering is a basis in all the spaces $B_{p,q}^s(Q)_{\boldsymbol{Z}}$ with $1 \leq p, q \leq \infty$, $0 < s < m$. Moreover, for*

$$f(\cdot) = \sum_{\boldsymbol{j}\geq \boldsymbol{n}(\boldsymbol{Z})} a_{\boldsymbol{j}}\, G_{\boldsymbol{j}}^{(m)}(\cdot\,;\boldsymbol{Z}),$$

letting $\sigma = s/d + 1/2 - 1/p$, we have

$$\|f\|_{p,q}^{(s)}(Q)_{\boldsymbol{Z}} \sim \left\{\sum_{\mu=0}^{\infty}\left[2^{\mu\sigma d}\Big(\sum_{\boldsymbol{j}\in N_\mu}|a_{\boldsymbol{j}}|^p\Big)^{1/p}\right]^q\right\}^{1/q}.$$

The constants in the relation $\sim$ depend on m, s and on d.

For an arbitrary d-cube Q, the function space $\mathcal{F}(Q)_{\boldsymbol{Z}}$ is defined as the image of $\mathcal{F}(I^d)_{\boldsymbol{Z}}$ under the linar mapping induced by the diffeomorphism between Q and I^d.

§4. Decomposition of Function Spaces over Smooth Manifolds

Let us start with the decomposition of M without boundary (for M with boundary cf. [16]). We say that M admits decomposition into d-cubes if for some N there are d-cubes $Q_1, \ldots, Q_N \subset M$ such that $\cup_{j\leq N} Q_j = M$ and if Φ_j is a diffeomorphism of $[0,1]^d$ onto Q_j, $1 \leq j \leq N$, then the set $\Phi_j^{-1}(\cup_{i<j} Q_i)$ is the union of some $(d-1)$-dimensional faces of $[0,1]^d$. The decomposition $Q_1, \ldots Q_N$, is said to be proper if the sets $\cup_{i\leq j} Q_i$ are proper for $j = 0, \ldots, N$. Now we have the following result whose proof depends very much on Morse theory (cf. Theorem 3.3 in [16]):

Theorem 12. *Let M be a compact d-manifold. Then M admits a proper decomposition.*

For the Sobolev $W_p^k(M)$ and Besov $B_{p,q}^s(M)$ spaces, we have the real interpolation formula for any integers l, r, real $s, l < s < r$, and for $1 \le p, q \le \infty$ (cf. [15])

$$B_{p,q}^s(M) = \left(W_p^l(M), W_P^r(M)\right)_{\theta,q}, \tag{44}$$

where $s = (1-\theta)l + \theta r$. We recall that for the standard Besov spaces, we have similar formula (cf. Proposition 5). Having a proper decomposition of M (Theorem 12), we would like to obtain a corresponding decomposition of $\mathcal{F}(M)$ into a direct sum of standard spaces $\mathcal{F}(Q_j)_{Z_j}$. Let $L_0(M)$ denote the space of all measurable functions (of equivalence classes) with the topology of convergence in measure.

Proposition 13. *Let $Q_1, \ldots, Q_N$ be a proper decomposition of M into d-cubes as in Theorem 12. Let μ be a smooth measure on M. Then, for any $m \ge 1$, one can construct continuous linear operators $P_1, \ldots, P_N$ in the space $L_0(M)$ with the following properties for $f \in L_0(M)$:*

$$\sum_{i \le N} P_i f = f, \tag{45}$$

$$P_i P_j f = 0, \qquad \text{if } 1 \le i \ne j \le N, \tag{46}$$

$$\chi_{Q_i} P_j f = P_i \chi_{Q_j} f, \qquad \text{if } 1 \le i < j \le N, \tag{47}$$

there is $C < \infty$ such that for all spaces $W_p^k(M)$, $0 \le k \le m, 1 \le p \le \infty$, for all $g \in W_p^k(M)$ and for $1 \le i \le N$ we have

$$\|P_i g\|_p^{(k)}(M) \le C \|g\|_p^{(k)}(M), \tag{48}$$

the adjoint operators (in the Hilbert space $L_2(M, \mu)$) $P_1^*, \ldots, P_N^*$ satisfy the analog of (48).

Proposition 13 implies the main result on decomposing the function spaces $\mathcal{F}(M)$ (see [16]), i.e.

Theorem 14. *Let $Q_1, \ldots, Q_N$ be a proper decomposition of M as in Theorem 12, and let $P_1, \ldots, P_N$ be the linear operators from Proposition 13. Then the formulae*

$$T_0 f = \sum_{i \le N} \chi_{Q_i} P_i f, \quad V_0 f = \sum_{i \le N} \chi_{Q_i} P_i^* f$$

define linear isomorphism of $L_0(M)$ onto itself, the inverse maps being, respectively,

$$S_0 f = \sum_{i \le N} P_i \chi_{Q_i} f, \quad U_0 f = \sum_{i \le N} P_i^* \chi_{Q_i} f.$$

Moreover, if $\mathcal{F}$ denotes W_p^k, $0 \le k \le m$, $1 \le p \le \infty$, then T_0, V_0 induce linear topological isomorphism

$$T : \mathcal{F}(M) \to \bigoplus_{i \le N} \mathcal{F}(Q_i)_{Z_i}, \quad V : \mathcal{F}(M) \to \bigoplus_{i \le N} \mathcal{F}(Q_i)_{Z_i'},$$

where $Z_i = Q_i \cup_{i<j} Q_j$ and $Z_i' = Q_i \cup_{j>i} Q_j$.

Corollary 15. *The assertions of Theorem 14 remain true for $\mathcal{F} = W_p^k$, $B_{p,q}^s$ with $|k| \leq m, |s| < m, 1 \leq p, q \leq \infty$. Moreover, there is now an obvious extension of Theorem 10 and Corollary 11 to $\mathcal{F}(M)$ for all these spaces.*

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